Problem 3.19

Use Equation 3.73 (or Problem 3.18 (c) and (d)) to show that:

(a) For any (normalized) wave packet representing a free particle ($V(x) = 0$), $\langle x \rangle$ moves at constant velocity (this is the quantum analog to Newton's first law). *Note:* You showed this for a *gaussian* wave packet in Problem 2.42, but it is completely general.

(b) For any (normalized) wave packet representing a particle in the harmonic oscillator potential $\left(V(x) = \frac{1}{2}m\omega^2 x^2\right)$, $\langle x \rangle$ oscillates at the classical frequency. *Note:* You showed this for a particular *gaussian* wave packet in Problem 2.49, but it is completely general.

Solution

Start with the result from part (c) of Problem 3.18.

$$m\frac{d\langle x \rangle}{dt} = \langle p \rangle$$

Differentiate both sides with respect to t.

$$\frac{d}{dt}\left(m\frac{d\langle x \rangle}{dt}\right) = \frac{d}{dt}\langle p \rangle$$

Use the result from part (d) of Problem 3.18.

$$m\frac{d^2\langle x \rangle}{dt^2} = \left\langle -\frac{dV}{dx} \right\rangle \tag{1}$$

This is the quantum analog to Newton's second law.

Part (a)

Suppose there's a free particle represented by a wave packet that satisfies $\langle x \rangle = x_0$ at $t = 0$ and $d\langle x \rangle/dt = v_0$ at $t = 0$. Equation (1) becomes

$$\begin{aligned} m\frac{d^2\langle x \rangle}{dt^2} &= \left\langle -\frac{d}{dx}(0) \right\rangle \\ &= \langle 0 \rangle \\ &= \langle \Psi \,|\, 0 \,|\, \Psi \rangle \\ &= \int_{-\infty}^{\infty} \Psi^*(x,t)(0)\Psi(x,t)\,dx \\ &= 0. \end{aligned}$$

Divide both sides by m.

$$\frac{d^2\langle x \rangle}{dt^2} = 0$$

Integrate both sides with respect to t.

$$\frac{d\langle x \rangle}{dt} = C_1$$

Integrate both sides with respect to t once more.

$$\langle x \rangle = C_1 t + C_2$$

Apply the two initial conditions to determine C_1 and C_2.

$$\text{At } t = 0: \qquad \langle x \rangle = C_2 = x_0$$

$$\text{At } t = 0: \qquad \frac{d\langle x \rangle}{dt} = C_1 = v_0$$

Therefore,

$$\langle x \rangle = x_0 + v_0 t,$$

which indicates the wave packet moves with constant velocity v_0.

Part (b)

Suppose there's a particle represented by a wave packet that satisfies $\langle x \rangle = x_0$ at $t = 0$ and $d\langle x \rangle / dt = v_0$ at $t = 0$ in the harmonic oscillator potential. Equation (1) becomes

$$\begin{aligned} m\frac{d^2\langle x \rangle}{dt^2} &= \left\langle -\frac{d}{dx}\left(\frac{1}{2}m\omega^2 x^2\right)\right\rangle \\ &= \langle -m\omega^2 x \rangle \\ &= \langle \Psi \,|\, -m\omega^2 x \,|\, \Psi \rangle \\ &= \int_{-\infty}^{\infty} \Psi^*(x,t)(-m\omega^2 x)\Psi(x,t)\,dx \\ &= -m\omega^2 \int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t)\,dx \\ &= -m\omega^2 \langle \Psi \,|\, \hat{x} \,|\, \Psi \rangle \\ &= -m\omega^2 \langle x \rangle. \end{aligned}$$

Divide both sides by m.

$$\frac{d^2\langle x \rangle}{dt^2} = -\omega^2 \langle x \rangle$$

The general solution for $\langle x \rangle$ can be written in terms of sine and cosine.

$$\langle x \rangle = C_3 \cos\omega t + C_4 \sin\omega t$$

Use the new arbitrary constants, A and δ, in place of C_3 and C_4.

$$\begin{aligned} \langle x \rangle &= A\cos\delta\cos\omega t + A\sin\delta\sin\omega t \\ &= A\cos(\omega t - \delta) \end{aligned}$$

Differentiate this solution with respect to t.

$$\frac{d\langle x \rangle}{dt} = -A\omega\sin(\omega t - \delta)$$

Apply the two initial conditions to determine A and δ.

$$\text{At } t = 0: \qquad \langle x \rangle = A\cos\delta = x_0$$

$$\text{At } t = 0: \qquad \frac{d\langle x \rangle}{dt} = A\omega\sin\delta = v_0$$

Square both sides of each equation and add them together to eliminate δ.

$$(A\cos\delta)^2 + (A\sin\delta)^2 = x_0^2 + \left(\frac{v_0}{\omega}\right)^2$$

$$A^2 = \frac{\omega^2 x_0^2 + v_0^2}{\omega^2}$$

$$A = \frac{\sqrt{\omega^2 x_0^2 + v_0^2}}{\omega}$$

Divide the respective sides instead to eliminate A.

$$\frac{A\sin\delta}{A\cos\delta} = \frac{\frac{v_0}{\omega}}{x_0}$$

$$\tan\delta = \frac{v_0}{\omega x_0}$$

$$\delta = \tan^{-1}\left(\frac{v_0}{\omega x_0}\right) + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Therefore,

$$\langle x \rangle = \frac{\sqrt{\omega^2 x_0^2 + v_0^2}}{\omega}\cos(\omega t - \delta),$$

which indicates the wave packet oscillates with an angular frequency of ω.